UDC 539.375

INTEGRAL EQUATIONS OF THE PLANE PROBLEM OF CRACK THEORY

PMM Vol. 38, № 4, 1974, pp. 728-737
A. P. DATSYSHIN and M. P. SAVRUK
(L'vov)
(Received April 16, 1973)

The plane problem of elasticity theory for a body weakened by a system of arbitrarily oriented rectilinear cracks has been reduced to a system of singular integral equations. The following cases have been considered: a system of cracks in infinite and semi-infinite plates, a system of cracks in an infinite plane with a circular hole, a periodic and doubly-periodic system of cracks of arbitrary orientation in an unbounded plate, and a periodic system of cracks in a semi-infinite plane. The analytical solution of the equations obtained can be found by the perturbation method when the cracks are far from each other and from the domain boundaries; in other cases their solution can be found numerically.

1. In an elastic isotropic plane connected to the Cartesian xOy-coordinate system, let there be N slits (cracks) of length $2a_k (k = 1, 2, \ldots, N)$. The centers O_k of the cracks are determined by the coordinates $z_k^0 = x_k^0 + iy_k^0$. The origins of local $x_kO_ky_k$-coordinate systems are placed at the points O_k. The O_kx_k-axes coincide with the slit lines and make the angles α_k with the Ox-axis. The edges of the cracks are loaded by the self-equilibrating forces

$$p_k(x_k) = N_k^+ - iT_k^+ = N_k^- - iT_k^-, \quad |x_k| \leqslant a_k, \ k = 1, 2, \ldots, N \qquad (1.1)$$

Let us first consider the problem of determining the stresses in an unbounded elastic plane with one crack $|x_k| \leqslant a_k$, $y_k = 0$ and the displacement discontinuity $g_k(x_k)$ given thereon. The Muskhelishvili [1] stress functions $\Phi(z_k)$ and $\Psi(z_k)$ for the problem mentioned are in the $x_kO_ky_k$-coordinate system [2, 4]

$$\Phi(z_k) = \frac{1}{2\pi} \int_{-a_k}^{a_k} \frac{g_k'(t)\, dt}{t - z_k}, \qquad z_k = e^{-i\alpha_k}(z - z_k^\circ) \qquad (1.2)$$

$$\Psi(z_k) = \frac{1}{2\pi} \int_{-a_k}^{a_k} \left[\frac{\overline{g_k'(t)}}{t - z_k} - \frac{t g_k'(t)}{(t - z_k)^2}\right] dt$$

By virtue of the linearity of the problem, the functions

$$\Phi_1(z) = \frac{1}{2\pi} \sum_{k=1}^{N} \int_{-a_k}^{a_k} \frac{g_k'(t)\, dt}{t - z_k}, \qquad T_k = te^{i\alpha_k} + z_k^\circ \qquad (1.3)$$

$$\Psi_1(z) = \frac{1}{2\pi} \sum_{k=1}^{N} e^{-2i\alpha_k} \int_{-a_k}^{a_k} \left[\frac{\overline{g_k'(t)}}{t - z_k} - \frac{\overline{T}_k e^{i\alpha_k}}{(t - z_k)^2} g_k'(t)\right] dt$$

obtained by superposition of the stress functions (1.2) for the isolated cracks, describe the state of stress of an elastic plane caused by the displacement discontinuities $g_k(x_k)$ at the N segments $|x_k| \leqslant a_k$, $y_k = 0$ $(k = 1, 2, \ldots, N)$. Let us determine the stresses on the line $O_n x_n$ due to the displacement discontinuities $g_k(x_k)$ $(k = 1, 2, \ldots, N)$

$$N_n - iT_n = \overline{\Phi_1(z_n)} + z_n\overline{\Phi_1'(z_n)} + \overline{\Psi_1(z_n)} + \Phi_1(z_n) = \tag{1.4}$$

$$\frac{1}{\pi}\sum_{k=1}^{N}\int_{-a_k}^{a_k}[K_{nk}(t, x_n)\, g_k'(t) + L_{nk}(t, x_n)\,\overline{g_k'(t)}]\, dt$$

Here

$$K_{nk}(t, x) = \frac{e^{i\alpha_k}}{2}\left[\frac{1}{T_k - X_n} + \frac{e^{-2i\alpha_n}}{\overline{T}_k - \overline{X}_n}\right], \quad X_n = xe^{i\alpha_n} + z_n^\circ \tag{1.5}$$

$$L_{nk}(t, x) = \frac{e^{-i\alpha_k}}{2}\left[\frac{1}{\overline{T}_k - \overline{X}_n} - \frac{T_k - X_n}{(\overline{T}_k - \overline{X}_n)^2}e^{-2i\alpha_n}\right]$$

Equating the stresses (1.4) to the stresses (1.1) given on the edges of the cracks, we obtain a system of N singular integral equations in the unknown functions $g_k'(x)$ [5, 6]

$$\sum_{k=1}^{N}\int_{-a_k}^{a_k}[g_k'(t)\, K_{nk}(t, x) + \overline{g_k'(t)}\, L_{nk}(t, x)]\, dt = \pi p_n(x) \tag{1.6}$$

$$|x| \leqslant a_n, \quad n = 1, 2, \ldots, N$$

(Here and henceforth, for convenience, the subscript in the x_n is omitted). The kernels (1.5) of this system of equations are regular, with the exception of the case $n = k$ when $K_{nk}(t, x)$ goes over into the Cauchy singular kernel.

2. Let us consider the centers of the cracks to be on the Ox-axis, the spacings between the centers of adjacent cracks to be constant and equal to $d(z_k^0 = kd,\ k = 0, \pm 1, \pm 2, \ldots)$, and the lengths and slopes of the cracks to be identical $(a_k = a, \alpha_k = \alpha)$. Under the assumption that the same load $(p_k(x_k) = p(x_k))$ is applied to all the cracks and the number of cracks tends to infinity, we obtain a periodic system of cracks of arbitrary orientation in an infinite plane. Hence $g_k'(x_k) = g'(x_k)$. After summation we find from (1.3)

$$\Phi_2(z) = \frac{e^{i\alpha}}{2d}\int_{-a}^{a}\operatorname{ctg}\frac{\pi}{d}(te^{i\alpha} - z)\, g'(t)\, dt \tag{2.1}$$

$$\Psi_2(z) = \frac{e^{i\alpha}}{2d}\int_{-a}^{a}\left\{\overline{g'(t)}\, e^{-2i\alpha}\operatorname{ctg}\frac{\pi}{d}(te^{i\alpha} - z) - \right.$$

$$\left.\left[\operatorname{ctg}\frac{\pi}{d}(te^{i\alpha} - z) + \frac{\pi}{d}(t - te^{2i\alpha} + ze^{i\alpha})\operatorname{cosec}^2\frac{\pi}{d}(te^{i\alpha} - z)\right] g'(t)\right\} dt$$

Having determined the stresses on the line of any of the cracks, with center at the point O, say, by means of these functions and having equated them to the given load (1.1), we obtain a singular integral equation in the unknown function $g'(x)$

$$\int_{-a}^{a}[g'(t)\, K_1(t - x) + \overline{g'(t)} L_1(t - x)]\, dt = \pi p(x), \quad |x| \leqslant a \tag{2.2}$$

Here

$$K_1(x) = \frac{\pi}{2d}\left(e^{i\alpha}\operatorname{ctg}\frac{\pi x e^{i\alpha}}{d} + e^{-i\alpha}\operatorname{ctg}\frac{\pi x e^{-i\alpha}}{d}\right) \tag{2.3}$$

$$L_1(x) = \frac{\pi}{2d}(e^{-i\alpha} - e^{-3i\alpha})\left(\operatorname{ctg}\frac{\pi x e^{-i\alpha}}{d} - \frac{\pi x e^{-i\alpha}}{d}\operatorname{cosec}^2\frac{\pi x e^{-i\alpha}}{d}\right)$$

In the case of an infinite series of colinear cracks $(\alpha = 0)$, we arrive at the equation

$$\frac{1}{d}\int_{-a}^{a} g'(t)\operatorname{ctg}\frac{\pi(t-x)}{d}\,dt = p(x), \quad |x| \leqslant a \tag{2.4}$$

considered in [7] for a normal load on the edges of the crack $(\operatorname{Im} p(x) = 0)$. The exact solution of (2.4) is

$$g'(x) = \left(d\cos^2\frac{\pi x}{d}\sqrt{\operatorname{tg}^2\frac{\pi a}{d} - \operatorname{tg}^2\frac{\pi x}{d}}\right)^{-1} \times \tag{2.5}$$

$$\int_{-a}^{a}\sqrt{\operatorname{tg}^2\frac{\pi a}{d} - \operatorname{tg}^2\frac{\pi t}{d}}\left(\operatorname{tg}\frac{\pi x}{d} - \operatorname{tg}\frac{\pi t}{d}\right)^{-1} p(t)\,dt$$

For $\alpha = \pi / 2$ we arrive, from (2.2), at an integral equation for an infinite series of parallel cracks, obtained for a symmetric $(\operatorname{Im} p(x) = 0)$ [8, 9] and antisymmetric $(\operatorname{Re} p(x) = 0)$ [10] load.

3. Let us consider an unbounded elastic plane weakened by a doubly-periodic system of cracks of arbitrary orientation. Let the centers of the cracks be at the vertices of the period parallelograms, i.e. at the points $P = m\omega_1 + n\omega_2$ $(m,\ n = 0, \pm 1, \pm 2, \ldots)$, where ω_1 and ω_2 are the fundamental periods $(\operatorname{Im}\omega_1 = 0,\ \operatorname{Im}(\omega_2 / \omega_1) > 0)$.

The complex potentials of the elasticity theory problem for the domain mentioned can be obtained analogously to the case of the periodic problem by setting $a_k = a$, $\alpha_k = \alpha$, $z_k^\circ = P$, $p_k(x_k) = p(x_k)$ in (1.3). Then $g_k'(x_k) = g'(x_k)$ and we obtain from (1.3)

$$\Phi_3(z) = \frac{e^{i\alpha}}{2\pi}\int_{-a}^{a}\zeta(te^{i\alpha} - z)\,g'(t)\,dt + A \tag{3.1}$$

$$\Psi_3(z) = \frac{1}{2\pi}\int_{-a}^{a}\{e^{-i\alpha}\zeta(te^{i\alpha} - z)\overline{g'(t)} +$$

$$[e^{i\alpha}\rho_1(te^{i\alpha} - z) - t\rho(te^{i\alpha} - z)]\,g'(t)\}\,dt + Be^{-2i\alpha}$$

Here ζ is the Weierstrass zeta function, $\rho(z)$ is the Weierstrass elliptic function, and $\rho_1(z)$ is a special meromorphic function [11]. At congruent points these functions satisfy the relationships [12]

$$\rho(z + \omega_\nu) - \rho(z) = 0, \qquad \zeta(z + \omega_\nu) - \zeta(z) = \delta_\nu \tag{3.2}$$

$$\rho_1(z + \omega_\nu) - \rho_1(z) = \overline{\omega}_\nu\rho(z) + \gamma_\nu$$

$$\delta_\nu = 2\zeta\left(\frac{\omega_\nu}{2}\right), \quad \gamma_\nu = 2\rho_1\left(\frac{\omega_\nu}{2}\right) - \overline{\omega}_\nu\rho\left(\frac{\omega_\nu}{2}\right), \qquad \nu = 1, 2$$

$$\delta_1\omega_2 - \delta_2\omega_1 = 2\pi i, \qquad \gamma_2\omega_1 - \gamma_1\omega_2 = \delta_1\overline{\omega}_2 - \delta_2\overline{\omega}_1$$

In order to avoid divergent sums, the unknown constants A and B, which can be determined from the static conditions [11], were introduced to obtain the complex potentials (3.1). To this end, let us consider the principal vector of all the forces acting along the arbitrary path CD connecting two congruent points of the plane. The expression for the principal vector is [1]

$$X + iY = -iq(z)|_C^D = -i[\varphi_3(z) + z\overline{\Phi_3(z)} + \psi_3(z)]_C^D \qquad (3.3)$$

$$(\varphi_3'(z) = \Phi_3(z),\ \psi_3'(z) = \Psi_3(z))$$

A self-equilibrated load acts on each crack, hence, the principal vector of all the forces along the arc CD is zero, i.e.

$$q(z + \omega_\nu) - q(z) = 0, \quad \nu = 1, 2 \qquad (3.4)$$

Using the equality

$$\int_{-a}^{a} g'(t)\,dt = 0, \qquad \zeta'(z) = -\rho(z) \qquad (3.5)$$

we find from (3.1) (C_1, C_2 are integration constants)

$$\varphi_3(z) = \frac{e^{2i\alpha}}{2\pi}\int_{-a}^{a} g(t)\,\zeta(te^{i\alpha} - z)\,dt + Az + C_1 \qquad (3.6)$$

$$\psi_3(z) = \frac{1}{2\pi}\int_{-a}^{a} \{[\overline{g(t)} + g(t)]\,\zeta(te^{i\alpha} - z) +$$

$$e^{i\alpha}g(t)\,[e^{i\alpha}\rho_1(te^{i\alpha} - z) - t\rho(te^{i\alpha} - z)]\}\,dt + Bze^{-2i\alpha} + C_2$$

Substituting the functions (3.6) into (3.3) and taking account of (3.2), we obtain a system of equations from (3.4)

$$(A + \bar{A})\omega_\nu + \bar{B}\bar{\omega}_\nu e^{2i\alpha} = \delta_\nu b + \bar{\gamma}_\nu b + \bar{\delta}_\nu(e^{-2i\alpha}b + \bar{b}e^{2i\alpha}), \qquad \nu = 1, 2 \quad (3.7)$$

$$b = \frac{e^{2i\alpha}}{2\pi}\int_{-a}^{a} g(t)\,dt$$

from which the constants $\operatorname{Re} A$ and B are determined. The quantity $\operatorname{Im} A$ naturally remains arbitrary.

The stress functions (3.1) correspond to the double-periodicity of the problem and satisfy all the requirements, with the exception of the boundary conditions on the edges of the cracks. Having satisfied the boundary condition on any crack, for example, one with center at the point O

$$\Phi(x_0) + \overline{\Phi(x_0)} + x_0\overline{\Phi'(x_0)} + \overline{\Psi(x_0)} = p(x_0) \qquad (3.8)$$

we arrive at a singular integral equation relative to the function $g'(x)$ (3.9)

$$\int_{-a}^{a} [g'(t)\,K_2(t - x) + \overline{g'(t)}\,L_2(t - x)]\,dt = \pi[p(x) - A - \bar{A} - \bar{B}], \quad |x| \leqslant a$$

$$K_2(x) = {}^1/_2[e^{i\alpha}\zeta(xe^{i\alpha}) + e^{-i\alpha}\bar{\zeta}(xe^{-i\alpha})]$$

$$L_2(x) = {}^1/_2[e^{-i\alpha}\bar{\zeta}(xe^{-i\alpha}) - xe^{-2i\alpha}\bar{\rho}(xe^{-i\alpha}) + e^{-3i\alpha}\bar{\rho}_1(xe^{-i\alpha})]$$

The constant $A + \bar{A} + \bar{B}$ in (3.9) is found easily from the system (3.7)

$$A + \bar{A} + \bar{B} = \frac{1}{\omega_1} [\delta_1 b + \bar{\gamma}_1 \bar{b} + \bar{\delta}_1 (e^{-2i\alpha} b + e^{2i\alpha} \bar{b})]$$

Here all the quantities are known, with the exception of the constant b which must be determined while solving the integral equation (3.9).

Let us note that the doubly-periodic problem of crack theory in [13] for the case when the fundamental period parallelogram is a rhombus, and the cracks are arranged along diagonals of the rhombus and tensile, constant intensity forces were applied to their edges, was reduced to a Fredholm integral equation of the second kind whose kernel is sufficiently complex.

4. Let there be $N + 1$ cracks of length $2a_k (k = 0, 1, \ldots, N)$ in an elastic plane. Let us assume the crack with subscript o to be on the Ox-axis ($\alpha_0 = 0$) with center at the origin of the xOy-coordinate system ($z_0^0 = 0$), while the remaining cracks are in the lower half-plane $y < 0$. Let us set $p_0(x) = 0$. Letting a_0 tend to infinity, we obtain a system of N cracks in an elastic half-plane with free edges. Equations (1.6) become

$$\int_{-\infty}^{\infty} \frac{g_0'(t)}{t - x} dt + \sum_{k=1}^{N} \int_{-a_k}^{a_k} [g_k'(t) K_{0k}(t, x) + \overline{g_k'(t)} L_{0k}(t, x)] dt = 0, \quad |x| < \infty \tag{4.1}$$

$$\sum_{k=1}^{N} \int_{-a_k}^{a_k} [g_k'(t) K_{nk}(t, x) + \overline{g_k'(t)} L_{nk}(t, x)] dt + \int_{-\infty}^{\infty} [g_0'(t) K_{n0}(t, x) + \overline{g_0'(t)} L_{n0}(t, x)] dt = \pi p_n(x) \tag{4.2}$$

$$|x| \leqslant a_n, \quad n = 1, 2, \ldots, N$$

Having determined the function $g_0'(x)$ from (4.1) and having substituted it into (4.2), we arrive after some manipulation at a system of N singular integral equations

$$\sum_{k=1}^{N} \int_{-a_k}^{a_k} [g_k'(t) M_{nk}(t, x) + \overline{g_k'(t)} N_{nk}(t, x)] dt = \pi p_n(x) \tag{4.3}$$

$$|x| \leqslant a_n, \quad n = 1, 2, \ldots, N$$

$$M_{nk}(t, x) = K_{nk}(t, x) + \frac{e^{i\alpha_k}}{2} \left\{ \frac{2}{X_n - \bar{T}_k} + \frac{e^{-2i\alpha_n}}{\bar{X}_n - T_k} + \frac{\bar{T}_k - T_k}{(X_n - T_k)^2} \left[1 + e^{-2i\alpha_n} - \frac{2e^{-2i\alpha_n}(X_n - T_k)}{\bar{X}_n - T_k} \right] \right\}$$

$$N_{nk}(t, x) = L_{nk}(t, x) + \frac{e^{-i\alpha_k}}{2} \left[\frac{T_k - \bar{T}_k}{(X_n - \bar{T}_k)^2} + \frac{1}{\bar{X}_n - T_k} - \frac{X_n - T_k}{(\bar{X}_n - T_k)^2} e^{-2i\alpha_n} \right]$$

For $N = 1$ and $\alpha_1 = \pi / 2$ $\operatorname{Im} p_1(x) = 0$, we arrive from (4.3) at an integral equation [14]. Let us note that integral equations of the elasticity theory problem for a strip weakened by a system of arbitrarily arranged cracks, can be obtained in a similar manner.

5. Let us assume the centers of all the cracks to be on a line parallel to the boundary of the half-plane, and the spacing between the centers of adjacent cracks to be identically equal to d, i.e. $z_k^\circ = x_k^\circ + iy_k^\circ = kd - ih\ (k = 0, \pm 1, \pm 2, \ldots)$, the lengths and slopes of the cracks to be equal $(a_k = a,\ \alpha_k = \alpha)$, and an identical load $p_k(x_k) = p(x_k))$ to be applied to all the cracks. Letting the number of cracks tend to infinity, we obtain a periodic system of cracks of arbitrary orientation $(g_k(x_k) = g(x_k))$ in the half-plane. We find the integral equation of such a problem analogously to the case of an infinite plate with a periodic system of cracks

$$\int_{-a}^{a} [g'(t) M(t, x) + \overline{g'(t)} N(t, x)]\, dt = \pi p(x), \quad |x| \leqslant a \tag{5.1}$$

Here

$$M(t, x) = K_1(t - x) + \frac{\pi e^{i\alpha}}{2d}\left\{\operatorname{ctg} X(t, x) + e^{-2i\alpha} \operatorname{ctg} \overline{X}(t, x) - \right. \tag{5.2}$$
$$\frac{2\pi i}{d}(t \sin\alpha - h) \operatorname{cosec}^2 \overline{X}(t, x)\left[1 - e^{-2i\alpha} - \frac{4\pi i e^{-2i\alpha}}{d} \times \right.$$
$$\left.\left.(x \sin\alpha - h) \operatorname{ctg} \overline{X}(t, x)\right]\right\}$$
$$N(t, x) = L_1(t - x) + \frac{\pi e^{-i\alpha}}{2d}\left[\frac{2\pi i}{d}(t \sin\alpha - h) \operatorname{cosec}^2 X(t, x) + \right.$$
$$\left.(1 - e^{-2i\alpha}) \operatorname{ctg} \overline{X}(t, x) - \frac{2\pi i e^{-2i\alpha}}{d}(x \sin\alpha - h) \operatorname{cosec}^2 \overline{X}(t, x)\right],$$
$$X(t, x) = \frac{\pi}{d}(x e^{i\alpha} - 2ih - t e^{-i\alpha})$$

As is seen from (4.4) and (5.2), the kernels of the integral equations (4.3) and (5.1) consist of two members: the first agrees with the kernels of (1.5) and (2.3), and the second takes account of the influence of the edge of the half-plane.

6. Let us consider an elastic plane with circular holes of unit radius referred to the xOy-coordinate system with origin at the center of the hole. In this plane, let there be N arbitrarily oriented cracks whose edges are subjected to the forces (1.1). The contour of the hole γ is unloaded. Let us reduce the problem of determining the stress-strain state of such a domain to the solution of a system of integral equations.

Let us assume that all the cracks in an infinite plate without holes are outside the unit circle γ. Using the complex potentials (1.3) and the formula from [1], we find the combination of stresses $\sigma_r + i\tau_{r\theta}$ caused in the circle $\gamma(z = e^{i\theta} = \sigma)$ by displacement discontinuities $g_k(x_k)$ $(k = 1, 2, \ldots, N)$

$$\sigma_r + i\tau_{r\theta} = \Phi_1(z) + \overline{\Phi_1(z)} - e^{-2i\theta}[z\overline{\Phi_1'(z)} + \overline{\Psi_1(z)}] = \tag{6.1}$$
$$\frac{1}{2\pi}\sum_{k=1}^{N}\int_{-a_k}^{a_k}\left\{\left[\frac{1}{t - z_k} - \frac{e^{2i\alpha_k}}{\sigma^2(t - \bar{z}_k)}\right] g_k'(t) + \left[\frac{1}{t + \bar{z}_k} - \frac{\sigma - T_k e^{i\alpha_k}}{\sigma^2(t - \bar{z}_k)^2}\right]\overline{g_k'(t)}\right\} dt$$

We solve the auxiliary problem for an elastic plane with a circular hole of unit radius on whose contour normal and shear stresses are given which are equal in magnitude but opposite in sign to the stresses (6.1). The stress-strain state in this case is characterized by the functions [1]

$$\Phi_4(z) = \frac{1}{2\pi i z}\int_{\gamma} \frac{\sigma(\sigma_r + i\tau_{r\theta})}{\sigma - z}\, d\sigma + \frac{a_1}{z}$$

$$\Psi_4(z) = -\frac{1}{2\pi iz}\int_\gamma \frac{\sigma_r - i\tau_{r\theta}}{\sigma(\sigma - z)} d\sigma + \frac{\Phi_4(z)}{z^2} - \frac{\Phi_4'(z)}{z} + \frac{a_1'}{z}$$

We should here set $a_1 = a_1' = 0$, since the principal vector of the forces applied to the contour γ is zero. Substituting the stresses (6.1) here and reversing the order of integration, we find after evaluating integrals

$$\Phi_4(z) = \frac{1}{2\pi}\sum_{k=1}^{N}\int_{-a_k}^{k}\left\{-\frac{\overline{T}_k e^{i\alpha_k}}{1-\overline{T}_k z} g_k'(t) + \right.$$

$$\left.\left[\frac{1-e^{-i\alpha_k}}{z\overline{T}_k^{\,2}} + \frac{1}{\overline{T}_k(1-\overline{T}_k z)} + \frac{ze^{-i\alpha_k} - T_k}{(1-\overline{T}_k z)^2}\right] e^{-i\alpha_k}\overline{g_k'(t)}\right\} dt$$

$$\Psi_4(z) = \frac{1}{2\pi z}\sum_{k=1}^{N}\int_{-a_k}^{a_k}\left\{\left[\frac{1}{T_k z} - \frac{\overline{T}_k}{z(1-\overline{T}_k z)} + \frac{\overline{T}_k^{\,2}}{(1-\overline{T}_k z)^2}\right]\times\right.$$

$$e^{i\alpha_k} g_k'(t) + \left[\frac{1}{z\overline{T}_k} + \frac{2(1-e^{-i\alpha_k})}{z^2\overline{T}_k^{\,2}} - \frac{z+T_k}{z(1-z\overline{T}_k)^2} - \right.$$

$$\left.\left.\frac{2\overline{T}_k(ze^{-i\alpha_k} - T_k)}{(1-\overline{T}_k z)^3}\right] e^{-i\alpha_k}\overline{g_k'(t)}\right\}$$

The elastic equilibrium of an infinite plate with a free circular boundary and displacement jumps $g_k(x_k)$ $(k = 1, 2, \ldots N)$ given on N segments $y_k = 0$, $|x_k| \leqslant a_k$ are determined by the complex potentials

$$\Phi_5(z) = \Phi_1(z) + \Phi_4(z), \qquad \Psi_5(z) = \Psi_1(z) + \Psi_4(z)$$

By requiring that the functions $\Phi_5(z)$ and $\Psi_5(z)$ satisfy the boundary conditions (1.1) on the edges of the cracks, we arrive at a system of N singular integral equations relative to the functions $g_k'(x_k)$

$$\sum_{k=1}^{N}\int_{-a_k}^{a_k}[R_{nk}(t,x)g_k'(t) + S_{nk}(t,x)\overline{g_k'(t)}]\,dt = \pi p_n(x) \tag{6.2}$$

$$|x| \leqslant a_n, \qquad n = 1, 2, \ldots, N$$

Here

$$R_{nk}(t,x) = K_{nk}(t,x) + \frac{e^{i\alpha_k}}{2}\left\{\frac{1-e^{i\alpha_k}}{\overline{X}_n T_k^{\,2}} - \frac{1}{X_n(1-\overline{T}_k X_n)} + \frac{1}{T_k(1-T_k\overline{X}_n)} + \right. \tag{6.3}$$

$$\frac{\overline{X}_n e^{i\alpha_k} - \overline{T}_k}{(1-T_k\overline{X}_n)^2} + e^{-2i\alpha_n}\left[\frac{2T_k(\overline{X}_n e^{i\alpha_k} - \overline{T}_k)}{\overline{X}_n(1-T_k\overline{X}_n)^3}(X_n\overline{X}_n - 1) + \frac{1-e^{i\alpha_k}}{\overline{X}_n^{\,3}T_k^{\,2}}(2 - X_n\overline{X}_n) + \right.$$

$$\left.\left.\frac{1}{\overline{X}_n^{\,2}T_k} + \frac{X_n(1+e^{i\alpha_k})}{(1-T_k\overline{X}_n)^2} - \frac{\dot{X}_n + \overline{T}_k}{\overline{X}_n^{\,2}(1-T_k\overline{X}_n)^2}\right]\right\}$$

$$S_{nk}(t,x) = L_{nk}(t,x) + \frac{e^{-i\alpha_k}}{2}\left\{\frac{1-e^{-i\alpha_k}}{X_n\overline{T}_k^{\,2}} - \frac{1}{\overline{X}_n(1-T_k\overline{X}_n)} + \frac{1}{\overline{T}_k(1-\overline{T}_k X_n)} + \right.$$

$$\left.\frac{X_n e^{-i\alpha_k} - T_k}{(1-\overline{T}_k X_n)^2} + e^{-2i\alpha_n}\left[\frac{T_k^{\,2}(1-X_n\overline{X}_n)}{\overline{X}_n(1-T_k\overline{X}_n)^2} + \frac{1}{\overline{T}_k\overline{X}_n^{\,2}} - \frac{T_k}{\overline{X}_n^{\,2}(1-T_k\overline{X}_n)}\right]\right\}$$

The second terms in (6.3) determine the perturbing influence of the circular hole. For $N = 1$, $\alpha_1 = 0$, we obtain an equation [3, 7, 15, 16] from (6.2) for the case of a radial crack in a plane with a circular hole. Let us also note that integral equations of the elasticity theory problem for a circular disc weakened by a system of arbitrarily arranged cracks can be obtained completely analogously.

7. By knowing the functions $g_k'(x)$ (or $g'(x)$), the stress-strain state of a plane with arbitrarily oriented cracks subjected to the forces (1.1) can be determined. In particular, we have the following formula for the stress intensity factor [17, 18] at the tips of any of the cracks

$$k_{1k}^{\pm} - ik_{2k}^{\pm} = \mp \lim_{x \to \pm a_k} \left[\frac{\sqrt{a_k^2 - x^2}}{\sqrt{a_k}} g_k'(x) \right], \qquad k = 1, 2, \ldots, N \qquad (7.1)$$

The quantities with the upper sign here refer to the right tips of the cracks and the lower to the left tips. Using the coefficients (7.1), the ultimate equilibrium state [18] of bodies with cracks can be investigated.

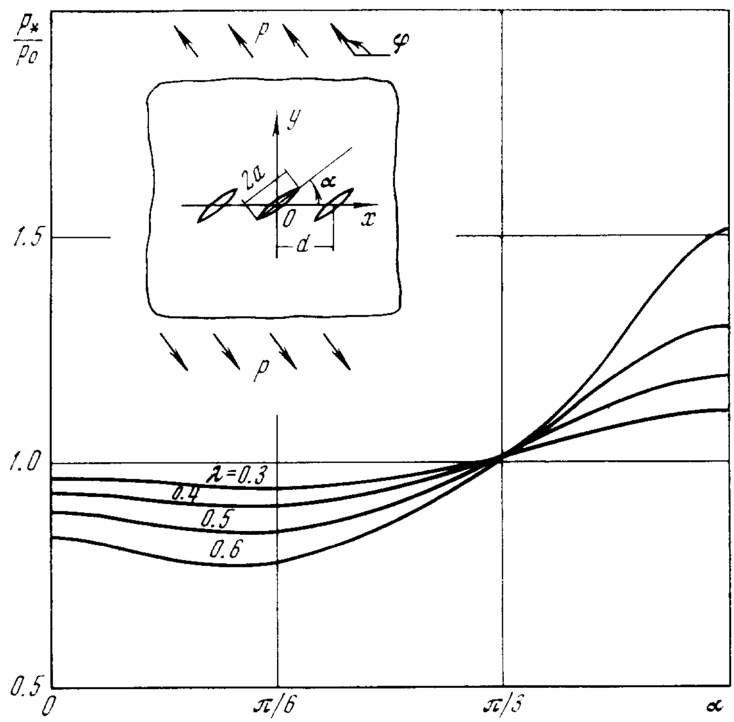

Fig. 1

As an illustration, let us examine the problem of the ultimate equilibrium of a plate with a periodic system of cracks which is subjected to biaxial tension at infinity by forces p and q in mutually perpendicular directions, where the forces p are directed at an angle φ to the Ox-axis. We will then have

$$p(x) = -s = -{}^1/_2 \left[(p + q) - (p - q)\, e^{2i(\varphi - \alpha)}\right]$$

in the integral equation (2.2).

The analytic solution of this equation is easily obtained [6] for small values of the dimensionless parameter $\lambda = 2a / d$. The stress intensity factors are determined by means of the formula

$$k_1^{\pm} - ik_2^{\pm} = \sqrt{a}\left\{s + \frac{\pi^2\lambda^2}{24}\left[s\cos 2\alpha + \bar{s}\left(e^{-2i\alpha} - e^{-4i\alpha}\right)\right] + \right.$$
$$\frac{\pi^4\lambda^4}{128}\left\{s\left[\frac{2}{9}\cos^2 2\alpha + \frac{1}{5}\cos 4\alpha + \frac{4}{9}(1-\cos 2\alpha)\right] + 2\bar{s}\left(e^{-4i\alpha} - e^{-6i\alpha}\right)\times\right.$$
$$\left.\left(\frac{1}{5} + \frac{2}{9}e^{2i\alpha}\cos 2\alpha\right)\right\} + \frac{\pi^6\lambda^6}{2^8 \cdot 3^2}\left\{s\left[\frac{5}{21}\cos 6\alpha + \frac{1}{6}\cos^3 2\alpha + \right.\right.$$
$$\frac{1}{4}\cos 2\alpha\cos 4\alpha + \frac{2}{5}e^{2i\alpha}(1-\cos 2\alpha) + \cos 2\alpha(1-\cos 2\alpha) +$$
$$\left.\frac{3}{5}e^{-2i\alpha}(1-\cos 2\alpha)\right] + \bar{s}\left(e^{-2i\alpha} - e^{-4i\alpha}\right)\left[\frac{5}{7}e^{-4i\alpha} + \frac{1}{2}e^{2i\alpha}\cos 2\alpha + \right.$$
$$\left.\left.\left.\frac{1}{4}\cos 4\alpha + \frac{1}{2}\cos^2 2\alpha + \frac{1}{3}(1-\cos 2\alpha)\right]\right\}\right\} + O(\lambda^8)$$

By using the Griffith-Irwin brittle fracture criterion and the hypothesis about the initial direction of crack propagation over areas with maximal normal stresses, we determine the magnitude of the crack extension forces. The dependence of the critical values p^* of the force p $(q = 0, \varphi = \pi / 2 + \alpha)$ on the angle of crack orientation α is represented in Fig. 1 for diverse values of λ (p_0 is the same value of the force p for the case of an isolated crack $(\lambda = 0)$).

In conclusion, let us note that the method proposed here to obtain the integral equations of the plane elasticity theory problem for a body with cracks, can be applied in other problems also, in particular, in bending problems of a plate with cracks, as well as in corresponding thermoelasticity problems.

REFERENCES

1. Muskhelishvili, N.I., Some Fundamental Problems of the Mathematical Theory of Elasticity. "Nauka", Moscow, 1966. (See also English translation, Groningen, Noordhoff, 1953).
2. Sherman, D.I., On an elasticity theory problem. Dokl. Akad. Nauk SSSR, Vol. 27, №9 1940.
3. Vitvitskii, P.M. and Leonov, M.Ia., Slip bands in inhomogeneous deformation of a plate. In: Questions of the Mechanics of a Real Solid, Pt. 1, UkrSSR Acad. of Sci. Press, Kiev, 1962.
4. Libatskii, L.L., Use of singular integral equations to determine the critical forces in plates with cracks. Fiziko-Khim. Mekh. Materialov, Vol. 1, №4, 1965.
5. Savruk, M.P. and Datsyshin, A.P., On the ultimate equilibrium state of a body weakened by a system of arbitrarily oriented cracks. In: Thermomechanical Methods of Fracturing Rock. Pt. 2. "Naukova Dumka", Kiev, 1972.
6. Datsyshin, A.P. and Savruk, M.P., A system of arbitrarily oriented cracks in elastic solids. PMM Vol. 37, №2, 1973.
7. Bueckner, H.F., Some stress singularities and their computation by means of integral equations. In: Boundary Problems of Differential Equations. Univ. of

Wisconsin Press, Madison, 1960.
8. Smith, E., The opening of parallel cracks by an applied tensile stress. Intern. J. Engng. Sci., Vol.4, №1, 1966.
9. Ichikawa, M., Ohashi, M. and Yokobori, T., Interaction between parallel cracks in an elastic solid and its effects on fracture. Repts. Res. Inst. Strength and Fract. Materials, Tohoku Univ., Vol.1, №1, 1965.
10. Savruk, M.P., Stresses in a plate with an infinite series of parallel cracks under antisymmetric load. Fiziko-Khim. Mekh. Materialov, Vol.8, №4, 1972.
11. Fil'shtinskii, L.A., Doubly-periodic problem of the theory of elasticity for an isotropic medium weakened by congruent groups of arbitrary holes. PMM Vol.36, №4, 1972.
12. Grigoliuk, E.I. and Fil'shtinskii, L.A., Perforated Plates and Shells. "Nauka", Moscow, 1970.
13. Kudriavtsev, B.A. and Parton, V.Z., First fundamental elasticity theory problem for a doubly-periodic system of cracks. In: Mechanics of a Continuous Medium and Kindred Analysis Problems. "Nauka", Moscow, 1972.
14. Sveklo, V.A., On a complex representation of solutions in plane elasticity theory. Inzh. Zh., Mekh.Tverd.Tela, №2, 1966.
15. Wigglesworth, L.A., Stress relief in a cracked plate. Matematika, Vol.5, №1, 1958.
16. Libatskii, L.L. and Baranovich, S.T., On a displacement discontinuity along rectilinear segments in a plate with a circular hole. Prikl. Mekh. №3, 1970.
17. Irwin, G.R., Handbuch der Physik, Bd.6, Springer, Berlin, 1958.
18. Panasiuk, V.V., Ultimate Equilibrium of Brittle Bodies with Cracks. "Naukova Dumka", Kiev, 1968.

Translated by M.D.F.

UDC 62.50

ON A LINEAR DIFFERENTIAL GAME OF EVASION

PMM Vol.38, №4, 1974, pp.738-742
V.M.RESHETOV
(Sverdlovsk)
(Received September 13, 1974)

For a linear controlled system we examine the evasion problem on an infinite semi-interval of time. The paper abuts the investigations in [1 – 5]. The solution is effected by the scheme of control with a leader [3, 4].

1. We examine a controlled system described by the vector differential equation

$$dx / dt = Ax + Bu + Cv, \quad u \in P, \quad v \in Q^{\alpha} \tag{1.1}$$

Here x is the k-dimensional phase coordinate vector, u and v are $r^{(1)}$- and $r^{(2)}$-dimensional vectors, respectively; A, B, C are matrices with constant coefficients of dimension $k \times k, k \times r^{(1)}, k \times r^{(2)}$, respectively; the first and second player's controls are constrained by the conditions indicated above, where P and Q are convex compacta in the corresponding vector spaces. The symbol Q^{α} denotes the closed Euclidean α-